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Time Correlation Functions of a One-Dimensional Infinite System

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We investigate the time evolution of a simple one-dimensional system with an infinite number of particles. We calculate some time correlation functions and show that they behave asymptotically as $1/\sqrt{t}$.

KEY WORDS: Infinite system; ergodic properties; correlation function.

1. INTRODUCTION

Let **Z** be a "one-dimensional" net formed by the intersection of two helices with opposite pitch wound on an infinite cylinder. The sites of **Z** are labeled by integers $m \in \mathbb{Z}$. Let $(\mathscr{X}, \mathbf{T}, \mu)$ be a system of infinitely many particles on the "lattice" **Z** with discrete velocities, namely, the particles are on the sites of **Z** and jump each unit of time in one of the four directions of the lattice, suffering collisions with each other in such a way that the particle number and the total momentum are conserved during a collision. This system is similar to the one introduced by Hardy *et al.*[1] This system is also essentially isomorphic to the one-dimensional hard-point system with two colors whose particles have integral positions and velocities v of unit magnitude $|v| = 1$ (Section 2.3).

In Section 2 we describe the model in detail and define the time evolution mapping **T** and the equilibrium measure μ.

In Section 3 we establish some fundamental properties of the system. We also show there that the system has a factor which is isomorphic to a Bernoulli system. So the system has positive entropy. The notion of fundamental path introduced there plays an essential role in the later investigations.

In Section 4 we calculate the time correlation functions. Their orders of

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decay are $1/\sqrt{t}$, the same for all values of the density, and their coefficients depend analytically on the density.

The system is a K-system, as was shown by Aizenman.[16]

2. DESCRIPTION OF DYNAMICAL SYSTEM ($\mathscr{X}$, T, μ)

2.1. Let $\mathbf{Z}$ be a "one-dimensional" net as is defined in the introduction. To define the time evolution of the system rigorously we represent $\mathbf{Z}$ as the quotient space of $\mathbb{Z}^2$ under the group $\mathbf{G}$ of shifts of $\mathbb{Z}^2$ generated by $\mathbf{g}$, where $\mathbf{g}$ is a translation of $\mathbb{Z}^2$ by the vector $(-1, 1)$. Let $\mathbf{P} = \{v_1, v_2, v_3, v_4\}$, where $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (-1, 0)$, and $v_4 = (0, -1)$.

Let

$$\mathscr{X} = \{X \mid X\colon\ \mathbf{Z} \times \mathbf{P} \to \{0, 1\}\}$$

For any $(a, v) \in \mathbf{Z} \times \mathbf{P}$, $X(a, v) = 1$ means that there is a particle with velocity v at the site a.

Let

$$\mathscr{X}_a = \{X_a = X_{|\{a\} \times \mathbf{P}} \mid X \in \mathscr{X}\}$$

Then naturally we have

$$\mathscr{X} \simeq \prod_{a \in \mathbf{Z}} \mathscr{X}_a$$

2.2. Now let us define the time evolution mapping $\mathbf{T}$ of the phase space $\mathscr{X}$. The mapping $\mathbf{T}$ is made up of the "free motion" $\mathbf{T}_0$ and the "collision" $\mathbf{C}$: $\mathbf{T} = \mathbf{C} \cdot \mathbf{T}_0$.

$\mathbf{T}_0$ is merely a translation of $\mathscr{X}$:

$$(\mathbf{T}_0 X)(a, v) = X(a - v, v) \qquad \text{for} \quad \forall (a, v) \in \mathbf{Z} \times \mathbf{P}$$

$\mathbf{C}$ is defined by the interaction mapping ϕ of $\mathscr{X}_0$,

$$\mathscr{X}_0 = \{X_0 \mid X_0\colon\ \mathbf{P} \to \{0, 1\}\} \simeq 2^{\mathbf{P}}$$

[We identify a $X_0 \in \mathscr{X}_0$ with the subset $\{v \in \mathbf{P} \mid X_0(v) = 1\}$.]

We define ϕ by $\phi(\{v_1, v_3\}) = \{v_2, v_4\}$, $\phi(\{v_2, v_4\}) = \{v_1, v_3\}$, and otherwise ϕ maps identically. Using this mapping ϕ, we define the collision $\mathbf{C}$ as follows:

$$(\mathbf{C}X)_a = \phi(X_a), \qquad X_a \in \mathscr{X}_a \simeq \mathscr{X}_0$$

2.3. The obtained dynamical system ($\mathscr{X}$, $\mathbf{T}$, μ) with an invariant homogeneous probability measure μ (we define it later precisely) is similar to the one introduced by Hardy *et al.*[1]

For the following investigation it is conveneint to reformulate ($\mathscr{X}$, $\mathbf{T}$, μ) as follows.

Let

$$\mathbf{R} = \{s, d, \theta, \Theta\}$$

We identify each space $\mathscr{X}_a$ with the space $\mathbf{R} \times \mathbf{R}$ as follows:

Let f [resp. g] be a mapping from $2^{\{v_1, v_2\}}$ [resp. $2^{\{v_3, v_4\}}$] to $\mathbf{R}$ such that $f(\{v_1\}) = s, f(\{v_2\}) = d, f(\phi) = \theta, f(\{v_1, v_2\}) = \Theta$ [resp. $g(\{v_3\}) = d, g(\{v_4\}) = s, g(\phi) = \theta, g(\{v_3, v_4\}) = \Theta$]. We have

$$\mathscr{X}_a \simeq 2^{\{v_3, v_4\}} \times 2^{\{v_1, v_2\}} \underset{g \otimes f}{\simeq} \mathbf{R} \times \mathbf{R}$$

By these identifications the phase space $\mathscr{X}$ is identified with the space $(\mathbf{R} \times \mathbf{R})^{\mathbb{Z}}$, which we denote again by $\mathscr{X}$,

$$\mathscr{X} = (\mathbf{R} \times \mathbf{R})^{\mathbb{Z}} \ni \{(l_m, r_m)\}_{m \in \mathbb{Z}}$$

The time evolution mapping $\mathbf{T} = \mathbf{C} \cdot \mathbf{T}_0$ can be written as follows: For $X = \{(l_m, r_m)\}$

$$\mathbf{T}_0 X = \{(l_m', r_m')\}$$

where $(l_m', r_m') = (l_{m+1}, r_{m-1})$ and

$$\mathbf{C}X = \{(l_m'', r_m'')\}$$

where

$$(l_m'', r_m'') = \begin{cases} (s, d) & \text{if} \quad (l_m, r_m) = (d, s) \\ (d, s) & \text{if} \quad (l_m, r_m) = (s, d) \\ (l_m, r_m) & \text{otherwise} \end{cases}$$

The invariant probability measure μ that we consider is the product (Bernoulli) probability measure on $(\mathbf{R} \times \mathbf{R})^{\mathbb{Z}}$, $\mu = (\mu_0 \otimes \mu_0)^{\mathbb{Z}}$, where μ_0 is the measure on $\mathbf{R}$ such that $\mu_0(s) = \mu_0(d) = \rho(1 - \rho)$, $\mu_0(\theta) = (1 - \rho)^2$, $\mu_0(\Theta) = \rho^2$; here 4ρ denotes the density of the system.

Remark. The dynamical system $(\mathscr{X}, \mathbf{T}, \mu)$ represented in this way is isomorphic to the hard-point system with two colors whose particles have integral initial positions and velocities v of unit magnitude $|v| = 1$, if we identify the states θ and Θ.

3. FUNDAMENTAL PROPERTIES OF THE DYNAMICAL SYSTEM $(\mathscr{X}, \mathbf{T}, \mu)$

3.1. From the special properties of the time evolution mapping $\mathbf{T}$, we can easily observe the following properties.

1. If we identify the states s and d (this identification is compatible with

the time evolution **T**), then the obtained factor dynamical system is nothing but the ideal gas, that is, a system with no interaction. More precisely, we have the following:

Proposition 1. Let $h: \mathbf{R} \to \bar{\mathbf{R}} = \{\iota, \theta, \Theta\}$ be the mapping $h(s) = h(d) = \iota$, $h(\theta) = \theta$, $h(\Theta) = \Theta$, and $\bar{\mathscr{X}} = (\bar{\mathbf{R}} \times \bar{\mathbf{R}})^{\mathbf{Z}}$. Let $\bar{\mathbf{T}}$ be the transformation defined at any $\bar{X} = \{(\bar{l}_m, \bar{r}_m)\} \in \bar{\mathscr{X}}$ by

$$\bar{\mathbf{T}}\bar{X} = \{(\bar{l}_m', \bar{r}_m')\}$$

where $(\bar{l}_m', \bar{r}_m') = (\bar{l}'_{m+1}, \bar{r}'_{m-1})$. Then we have

$$\tilde{h}\cdot\mathbf{T} = \bar{\mathbf{T}}\cdot\tilde{h}$$

where $\tilde{h} = (h \otimes h)^{\mathbf{Z}}: \mathscr{X} \to \bar{\mathscr{X}}$.

Remark. The factor dynamical system $(\bar{\mathscr{X}}, \bar{\mathbf{T}}, \bar{\mu})$ $[\bar{\mu} = \tilde{h}(\mu)]$ is isomorphic to a Bernoulli system. Thus the system $(\mathscr{X}, \mathbf{T}, \mu)$ has a positive entropy.

2. Along the time evolution the states on the even sites (l_{2m}, r_{2m}) of the initial configuration do not interact with the states on the odd sites $(l_{2m'+1}, r_{2m'+1})$ of the initial configuration.

3.2. Now we consider the diagram of the time evolution of the configurations X of $\mathscr{X}$. Take an initial configuration X; we then describe the diagram of the time evolution of X as in Fig. 1. Here $X = \{(l_m, r_m)\}$ is taken to be for $m = -8,\ldots, 4, 6, 8$; $l_{-8} = r_{-6} = l_{-2} = l_0 = r_6 = s$; $l_{-6} = r_{-2} = r_2 = l_4 = l_8 = d$; and the other l_m and r_m are θ or Θ.

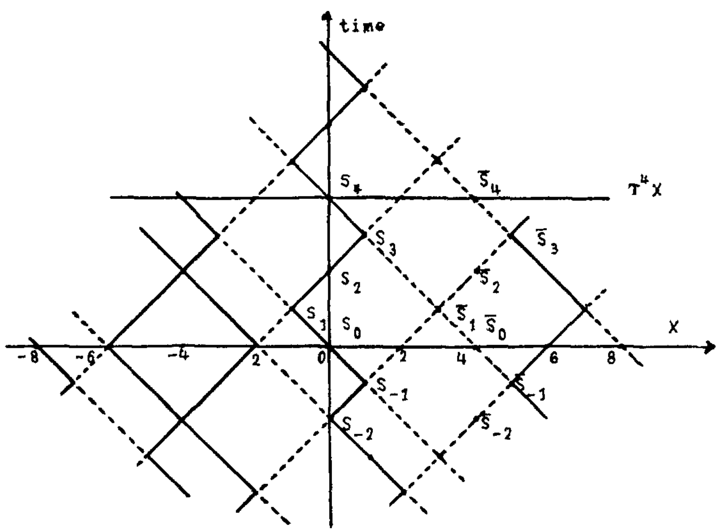

Fig. 1

In Fig. 1 solid (resp. dotted) lines show that along these lines the states s (resp. d) evolve. Note that we do not describe the diagram of the states that are initially on odd sites (l_{2m+1}, r_{2m+1}), because of the above property 2. The diagram of the states on odd sites and that of the states on even sites are mutually independent. Therefore from now on we consider only the states on even sites of the initial configurations. Similarly, the diagrams of the states θ and Θ are not described explicitly because of Proposition 1.

From these lines we pick up the paths S of X as in Fig. 1; more precisely we have the following:

Definition 1. Let $X \in \mathscr{X}$ and $\mathbf{T}^n X = \{(l_m^{(n)}, r_m^{(n)})\}$ $\forall n \in \mathbb{Z}$. A fundamental path of X is a sequence

$$S = \{..., s_{-1}, s_0, s_1, ...\}, \qquad s_i \in \mathbb{Z}$$

that satisfies (a) $\forall n \in \mathbb{Z}$, $h(l_{s_n}^{(n)}) = \iota$ or $h(r_{s_n}^{(n)}) = \iota$; (b) $s_{n+1} - s_n = \pm 1$; (c) $s_{n+1} - s_n$ changes sign, that is, $(s_{n+1} - s_n)(s_n - s_{n-1}) = -1$, if and only if $S_n = (s_n, n)$ is a point of intersection of the lines, $h(l_{s_n}^{(n)}) = h(r_{s_n}^{(n)}) = \iota$. Further, $s_{n+1} - s_n = 1$ (resp., $=-1$) if $h(r_{s_n}^{(n)}) = \iota$ [resp., $h(l_{s_n}^{(n)}) = \iota$].

We say that S passes through $l_m^{(n)}$ (resp. $r_m^{(n)}$) if $s_n = m$ and $s_{n+1} - s_n = -1$ (resp., $=1$).

One may regard a fundamental path as describing the motion of an "elementary excitation." The index of the sequence represents time.

Now we introduce an *order* among all the fundamental paths of X that pass through the even (or odd) sites, namely paths with an even (or odd) s_0.

Definition 2. For any two fundamental paths S and $\bar{S}$ of X, we call $S < \bar{S}$ if the path S lies on the left of the path $\bar{S}$, that is, if $s_n < \bar{s}_n$ for all n. Note that by Definition 1, if $S \nless \bar{S}$ then $\bar{S} < S$.

3.3. We have the following result:

Lemma 1. Along any fundamental path S of X alteration of the states s and d cannot take place. That is, let S pass through $x_{s_n}^{(n)}$, where $x = l$ or r; then $x_{s_{n_0}}^{(n)} = d$ (resp. $=s$) for some n_0 if and only if $x_{s_n}^{(n)} = d$ (resp. $=s$) for all n.

This lemma, which follows easily from the property of the collision $\mathbf{C}$, makes it possible to define the notion of the color of a path.

Definition 3. A fundamental path S of X has a *color* d (resp. s) if $x_{s_n}^{(n)} = d$ (resp. $=s$). We denote the color of S by $c(S)$.

It is also easy to prove (cf. Refs. 15 and 16) the following "Markov property" of the measure μ.

Lemma 2. For each $n \in \mathbb{Z}$ let $\mathbf{Q}_n$ be the partition of $\mathscr{X}$ according to whether there is a path passing through $l_0^{(n)}$ or $r_0^{(n)}$. Then the partitions $\mathbf{Q}_n$ are jointly independent.

This property is the main reason for the usefulness of our representation. Notice that the sequence formed by the colors of the paths that pass through $\{l_0^{(n)}, r_0^{(n)}\}$ is strongly restricted by consistency conditions. These result from Lemma 1 and the fact that the relative order of the paths is invariant in time.

4. TIME CORRELATION FUNCTIONS

4.1. Using properties established in Section 3, we can calculate explicitly the time correlation functions and as a consequence we can show that the system is mixing.

Now let us compute the correlation functions

$$\mathbf{C}_n(\mathbf{A}, \mathbf{B}; \rho) = \mu(\mathbf{A} \cap \mathbf{T}^{-n}\mathbf{B}) - \mu(\mathbf{A})\mu(\mathbf{B})$$

for arbitrary cylinder sets $\mathbf{A}$ and $\mathbf{B}$ of $\mathscr{X}$.

As an example, we compute the simplest one. Let, for instance,

$$\mathbf{A} = \{X = \{(l_m, r_m)\} | l_0 = s, r_0 = d\}$$

We will compute $\mathbf{C}_n(\mathbf{A}, \mathbf{A}; \rho)$.

Let $X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A}$. Let S^0 and S^1 be fundamental paths of X that pass through l_0 and r_0, respectively ($S^0 < S^1$). Let $\bar{S}^0$ and $\bar{S}^1$ be paths of X that pass through $l_0^{(2n)}$ and $r_0^{(2n)}$, respectively ($\bar{S}^0 < \bar{S}^1$). Note that $l_0^{(2n)} = s$ and $r_0^{(2n)} = d$, since $\mathbf{T}^{2n}X \in \mathbf{A}$ (see Fig. 2).

Let

$$L^n(X) = \#\{m | 0 < 2m < 2n, \quad h(l_{2m}(X)) = \iota\}$$
$$R^n(X) = \#\{m | 0 > -2m > -2n, \quad h(r_{-2m}(X)) = \iota\}$$
$$D^n(X) = L^n(X) - R^n(X)$$

Here $\#\{\cdots\}$ means the cardinality of the set $\{\cdots\}$.

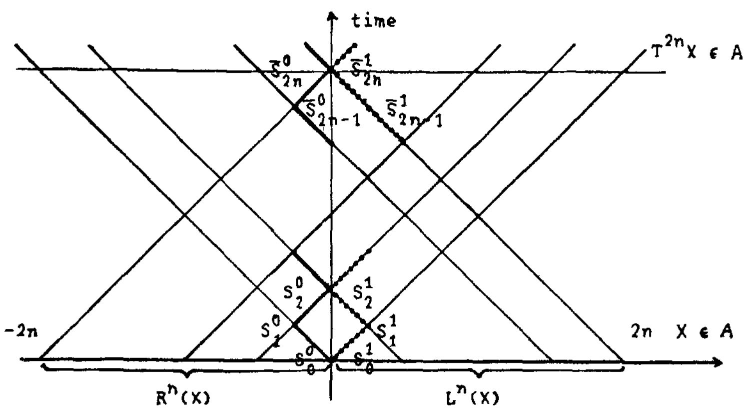

Fig. 2

Let

$$\mathbf{E}_- = \{X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} | D^n(X) < -1\}$$
$$\mathbf{E}_k = \{X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} | D^n(X) = k\}, \qquad k = -1, 0, 1$$
$$\mathbf{E}_+ = \{X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} | D^n(X) > 1\}$$

These sets $\mathbf{E}_-, \mathbf{E}_{-1}, \ldots, \mathbf{E}_+$ are mutually disjoint decompositions of $\mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A}$. These sets are characterized as follows:

$$\mathbf{E}_- = \{X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} | \bar{S}^1 < S^0\}$$
$$\mathbf{E}_{-1} = \{X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} | \bar{S}^1 = S^0\}$$
$$\mathbf{E}_0 = \{X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} | \bar{S}^0 = S^0, \bar{S}^1 = S^1\}$$
$$\mathbf{E}_1 = \{X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} | S^1 = \bar{S}^0\}$$
$$\mathbf{E}_+ = \{X \in \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} | S^1 < \bar{S}^0\}$$

Let us compute $\mu(E_0)$. We define

$$\mathbf{E} = \{h(l_0(X)) = h(r_0(X)) = h(l_{2n}(X)) = h(r_{-2n}(X)) = \iota\} \cap \{D^n(X) = 0\}$$

Then

$$\mathbf{E}_0 = \mathbf{A} \cap \mathbf{T}^{-2n}\mathbf{A} \cap \{D^n(X) = 0\}$$
$$= \mathbf{E} \cap \{c(S^0) = c(\bar{S}^0) = s, c(S^1) = c(\bar{S}^1) = d\}$$

But, under the condition $\mathbf{E}$, $S^0 = \bar{S}^0$ and $S^1 = \bar{S}^1$. Therefore

$$\mathbf{E}_0 = \mathbf{E} \cap \{c(S^0) = s, c(S^1) = d\}$$

Hence

$$\mu(\mathbf{E}_0) = \mu(\mathbf{E})\mu(\{c(S^0) = s, c(S^1) = d\}|\mathbf{E})$$
$$= (\tfrac{1}{2})^2\mu(\mathbf{E})$$
$$= (\tfrac{1}{2})^2\mu(\{h(l_0(X)) = \cdots = h(r_{-2n}(X)) = \iota\})\mu(\{D^n(X) = 0\})$$
$$= 2^2\mu(\mathbf{A})^2 P_0$$

Here we have used the independence of $\{h(l_0(X)) = \cdots = h(r_{-2n}(X)) = \iota\}$ and $\{D^n(X) = 0\}$.

The other probabilities can be computed similarly. They are given by

$$\mu(\mathbf{E}_-) = \mu(\mathbf{A})^2 P_{<-1}$$
$$\mu(\mathbf{E}_{-1}) = \mu(\mathbf{A})^2 \epsilon(S^0, \bar{S}^1) P_{-1}$$
$$\mu(\mathbf{E}_0) = \mu(\mathbf{A})^2 \epsilon(S^0, \bar{S}^0)\epsilon(S^1, \bar{S}^1) P_0$$
$$\mu(\mathbf{E}_1) = \mu(\mathbf{A})^2 \epsilon(S^1, \bar{S}^0) P_1$$
$$\mu(\mathbf{E}_+) = \mu(\mathbf{A})^2 P_{>1}$$

where we use the notations

$$P_k = \mu(\{D^n(X) = k\})$$

and

$$\epsilon(S, \bar{S}) = \begin{cases} 2 & \text{if } c(S) = c(\bar{S}) \\ 0 & \text{otherwise} \end{cases}$$

To compute $\mu(E_{-1})$ note that as $c(S^0) = s$ and $c(\bar{S}^1) = d$, so $E_{-1} = \phi$ and $\mu(E_{-1}) = 0$.

Finally, we get

$$\begin{aligned} \mathbf{C}_{2n}(\mathbf{A}, \mathbf{A}; \rho) &= \mu(\mathbf{E}_-) + \mu(\mathbf{E}_{-1}) + \cdots + \mu(\mathbf{E}_+) - \mu(\mathbf{A})^2 \\ &= \mu(\mathbf{A})^2(3P_0 - P_{-1} - P_1) \end{aligned}$$

Here we have used $P_{<-1} + P_{-1} + \cdots + P_{>1} = 1$.

The probabilities P_k,

$$\begin{aligned} P_k &= \mu(\{D^n(X) = k\}) \\ &= \sum_{j:0 \leqslant j, j+k \leqslant n-1} {}_{n-1}C_{j+k} p^j q^{n-1-j} p^{j+k} q^{n-1-j-k} \end{aligned}$$

where $p = 2\rho(1 - \rho)$ and $q = 1 - p$, appear in the calculation of the central limit theorem for the sum of a Markov chain. These asymptotic values are (see Refs. 7 and 8)

$$P_k \sim (4pq\pi n)^{-1/2} \qquad \text{as} \quad n \to \infty$$

Therefore

$$\mathbf{C}_{2n}(\mathbf{A}, \mathbf{A}; \rho) \sim (4pq\pi n)^{-1/2}\mu(\mathbf{A})^2$$

4.2. In the similar way we can get the following result:

Theorem 2. Let **A** and **B** be cylinder sets defined on even (or odd) sites simultaneously. The time correlation function of **A** and **B** behaves asymptotically as $1/\sqrt{n}$. More precisely,

$$\mathbf{C}_{2n}(\mathbf{A}, \mathbf{B}; \rho) \sim C(\mathbf{A}, \mathbf{B})(4pq\pi n)^{-1/2} \qquad \text{as} \quad n \to \infty$$

where $p = 2\rho(1 - \rho)$ and $q = 1 - p$. The coefficient $C(\mathbf{A}, \mathbf{B})$ depends only on **A** and **B** and can be calculated explicitly.

When $C(\mathbf{A}, \mathbf{B}) = 0$ the sign $\sim$ means

$$\mathbf{C}_{2n}(\mathbf{A}, \mathbf{B}; \rho) = o((4pq\pi n)^{-1/2})$$

We note that $C_{2n+1}(\mathbf{A}, \mathbf{B}; \rho) = 0$ and the general case is easily reduced to these cases by property 2 of Section 3.1.

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REFERENCES

1. J. Hardy, Y. Pomeau, and O. de Pazzis, *J. Math. Phys.* **14**:1746 (1973).
2. T. Niwa, *J. Math. Kyoto Univ.* **16**:209 (1976).
3. Ya. G. Sinai and K. Volkovysskv, *Funct. Anal. Appl.* **5**:19 (1971).
4. Ya. G. Sinai, *Funct. Anal. Appl.* **6**:41 (1972).
5. O. de Pazzis, *Comm. Math. Phys.* **22**:121 (1971).
6. O. E. Lanford III, *Acta Phys. Austriaca, Suppl. X* **1973**:619.
7. E. D. Pepper, *Ann. Math.* **28**:318 (1927).
8. R. L. Dobrushin, *Izv. Akad. Nauk SSSR* **17**:291 (1953).
9. V. I. Arnold and A. Avez, *Problèmes ergodiques de la mécanique classique* (Paris: Gauthiers-Villars, 1967).
10. W. Feller, *An Introduction to Probability Theory and Its Applications* (Wiley, New York, 1958), Vol. I.
11. V. A. Rohlin, *Uspekhi Mat. Nauk* **22**:3 (1967).
12. M. Aizenman, S. Goldstein, and J. L. Lebowitz, in *Dynamical Systems Theory and Applications* (Lecture Notes in Physics 38) (Springer, Berlin, 1975).
13. Dao-Quang-Tuyen and D. Szász, *Z. Wahr. verw. Gebiete* **31**:75 (1974).
14. D. Szász, Particle Systems with Collisions, Preprint of Math. Inst. Hungarian Academy of Sciences, 26 (1975).
15. M. Aizenman, S. Goldstein, and J. L. Lebowitz, *Comm. Math. Phys.* **39**:289 (1975).
16. M. Aizenman, Thesis, Yeshiva Univ. (1975).
17. O. W. Jepsen, *J. Math. Phys.* **6**:405 (1965).
18. J. L. Lebowitz and J. K. Percus, *Phys. Rev.* **155**:122 (1967).